

APPLICATIONS OF THE DUALITY BETWEEN THE COMPLEX MONGE-AMPÈRE EQUATION AND THE HELE-SHAW FLOW

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ABSTRACT. We give two applications of the the duality between the complex Homogeneous Monge-Ampère Equation (HMAE) and the Hele-Shaw flow. First, we prove existence of smooth boundary data for which the weak solution to the Dirichlet problem for the HMAE over $\mathbb{P}^1 \times \mathbb{D}$ is not twice differentiable at a given collection of points, and also examples that are not twice differentiable along a set of codimension one in $\mathbb{P}^1 \times \partial\mathbb{D}$. Second we discuss how to obtain explicit families of smooth geodesic rays in the space of Kähler metrics on \mathbb{P}^1 and on the unit disc \mathbb{D} that are constructed from an exhausting family of increasing smoothly varying simply connected domains.

1. INTRODUCTION

The purpose of this paper is to give two applications of previous work of the authors that describes a duality between a certain Dirichlet problem for the complex Homogeneous Monge-Ampère Equation (HMAE) and a free boundary problem in the plane called the Hele-Shaw flow [19]. First, for any finite set of points in $\mathbb{P}^1 \times \partial\mathbb{D}$, where $\mathbb{D} \subset \mathbb{C}$ denotes the unit disc, we give examples of smooth boundary data for which the weak solution to this Dirichlet problem over $\mathbb{P}^1 \times \mathbb{D}$ is not twice differentiable at these points. We also produce such examples that are not twice differentiable along a set of codimension one in $\mathbb{P}^1 \times \partial\mathbb{D}$. Second, we use this duality to produce families of regular solutions to this Dirichlet problem over the punctured disc \mathbb{D}^\times , giving explicit families of smooth geodesic rays in the space of Kähler metrics on \mathbb{P}^1 and on \mathbb{D} .

1.1. Regularity of the Dirichlet problem for the HMAE over the disc. The setup for the first application is as follows. Fix a chart $0 \in \mathbb{C} \subset \mathbb{P}^1$ with coordinate z and let ω denote the Fubini-Study form. Choose a Kähler potential $\phi \in C^\infty(\mathbb{P}^1)$, by which we mean $\omega + dd^c\phi$ is a Kähler form, and let $\pi_{\mathbb{P}^1} : \mathbb{P}^1 \times \mathbb{D} \rightarrow \mathbb{P}^1$ be the projection. Consider the envelope

$$\Phi := \sup \left\{ \begin{array}{l} \psi : \mathbb{P}^1 \times \mathbb{D} \rightarrow \mathbb{R} \cup \{-\infty\} : \psi \text{ is usc, } \pi_{\mathbb{P}^1}^* \omega + dd^c \psi \geq 0 \\ \text{and } \psi(z, \tau) \leq \phi(\tau z) \text{ for } (z, \tau) \in \mathbb{P}^1 \times \partial\mathbb{D} \end{array} \right\}, \quad (1)$$

which is the weak solution to the Dirichlet problem

$$\begin{aligned} \Phi(z, \tau) &= \phi(\tau z) \text{ for } (z, \tau) \in \mathbb{P}^1 \times \partial\mathbb{D}, \\ \pi_{\mathbb{P}^1}^* \omega + dd^c \Phi &\geq 0, \\ (\pi_{\mathbb{P}^1}^* \omega + dd^c \Phi)^2 &= 0. \end{aligned}$$

The following is a preliminary version of what we shall prove:

Theorem A. Let S be a union of finitely many points and non-intersecting smooth curve segments in $\mathbb{P}^1 \setminus \{0\}$. Then there exist Kähler potentials such that the above weak solution Φ to the HMAE is not twice differentiable at any $(\tau^{-1}z, \tau)$, $z \in S$, $|\tau| = 1$.

The question of regularity of solutions to the HMAE has a long history, and has proved to be a difficult problem that depends subtly on the boundary data (see, for example, Lempert [12], Bedford-Demailly [2] or Błocki [3]). As is well known, if $\overline{\mathbb{D}}$ is replaced by a closed annulus in \mathbb{C} , and \mathbb{P}^1 is replaced by any Kähler manifold X , the above Dirichlet problem with S^1 -invariant boundary data corresponds to finding a geodesic segment in the space of Kähler potentials on X (and similarly if $\overline{\mathbb{D}}$ is replaced by the punctured disc $\overline{\mathbb{D}}^\times$ it corresponds to finding a geodesic ray). The regularity of these geodesics has been of intense interest ever since this space was considered by Mabuchi [15] Semmes [22] and Donaldson [7]. However it is only since the relatively recent work of Lempert-Vivas [13] and Lempert-Darvas [14] that we have known that it is not always possible to join two potentials by a geodesic segment that lies in the class C^2 .

What we have here is similar in spirit, but is in a sense stronger than the result of Lempert-Vivas in that we are able to prescribe the location of the singular locus (which need not consist of isolated points), as well as see exactly how the regularity fails; we are not aware of any similar result in the theory of the HMAE in which this precise information about the weak solution is available, other than the toric case [20, 21].

What permits us to have such a good understanding of the singularities of Φ is the connection with the Hele-Shaw flow. To define this, suppose (X, ω) is a one-dimensional Kähler manifold, which we will usually take to be either \mathbb{P}^1 with its Fubini-Study form ω_{FS} , or the open unit disc $\mathbb{D} \subset \mathbb{C}$ with the Poincaré form ω_P . In the first case we use the convention that \mathbb{P}^1 has area one (of course the Poincaré metric on the disc has infinite area), so

$$V := \int_X \omega \in \{1, \infty\}.$$

The unit disc has the origin as a distinguished point, and when $X = \mathbb{P}^1$ we fix a point that we denote by $0 \in \mathbb{P}^1$.

Given any $\phi \in C^\infty(X)$ (if X is noncompact we also assume ϕ to be bounded) such that $\omega + dd^c \phi$ is a Kähler form, the Hele-Shaw flow consists of an increasing collection of sets

$$\Omega_t \subset X \text{ for } t \in (0, V)$$

such that Ω_t has area t with respect to $\omega + dd^c \phi$. It is defined by setting

$$\Omega_t := \{z \in X : \psi_t(z) < \phi(z)\}$$

where

$$\psi_t := \sup\{\psi : \psi \text{ is usc and } \psi \leq \phi \text{ and } \omega + dd^c \psi \geq 0 \text{ and } \nu_0(\psi) \geq t\}.$$

By this we mean the supremum is over all upper semicontinuous (usc) functions $\psi : X \rightarrow \mathbb{R} \cup \{-\infty\}$ with these properties, and $\nu_0(\psi)$ denotes the order of the logarithmic singularity (Lelong number) of ψ at $0 \in X$.

What is proved¹ in [19] is that this flow is intimately connected to the weak solution $\tilde{\Phi}$ to the Dirichlet problem for the complex HMAE on $X \times \overline{\mathbb{D}}^\times$ with boundary data the pullback of ϕ to $X \times \partial\mathbb{D}$ and a certain prescribed singularity at $(0, 0)$; in fact ψ_t is the Legendre transform of $\tilde{\Phi}$. Moreover there is a simple way to transform between Φ and $\tilde{\Phi}$, and thus each contain the same information as the Hele-Shaw flow (we shall recall this in more detail in Section 4).

¹Strictly speaking only the case that $X = \mathbb{P}^1$ is considered in [19], but the proof works for $X = \mathbb{D}$.

To state our first result more precisely we need to consider flows of sets that develop singularities in a particularly simple way. Let S be the union of finitely many points and non-intersecting smooth curve segments in $\mathbb{P}^1 \setminus \{0\}$.

Definition 1.1. We say that the Hele-Shaw for $\omega + dd^c\phi$ *develops tangency* along S if there exists a $T \in (0, 1)$ such that (1) Ω_t is smoothly bounded, simply connected and varies smoothly for $t < T$ and (2) Ω_T is simply connected and $\partial\Omega_T$ is the image of a smooth locally embedded curve intersecting itself tangentially precisely along S (see Figure 1).

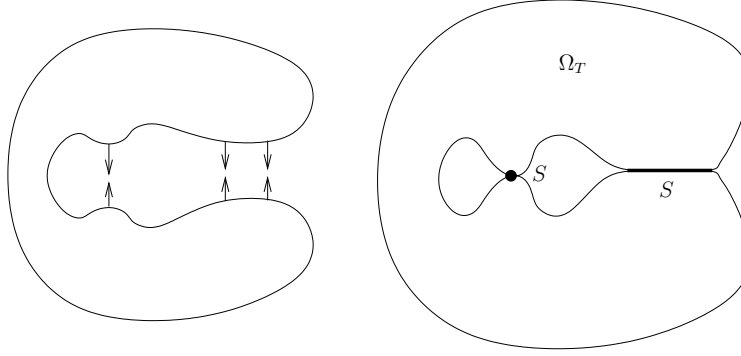


FIGURE 1. Developing tangency along S

Theorem B. Let $\phi \in C^\infty(\mathbb{P}^1)$ be a Kähler potential and suppose the Hele-Shaw for $\omega + dd^c\phi$ develops tangency along S . Then the weak solution Φ from (1) to the Dirichlet problem for the HMAE on $\mathbb{P}^1 \times \overline{\mathbb{D}}$ with boundary data $(z, \tau) \mapsto \phi(\tau z)$ is not twice differentiable at the points $(\tau^{-1}z, \tau)$, $z \in S$, $|\tau| = 1$.

We note that actually we know more, and from the discussion below it will be apparent that there is an explicit open set in $\mathbb{P}^1 \times \overline{\mathbb{D}}$ on which Φ is smooth. With more work it may be possible to describe precisely where (and how) Φ fails to be twice differentiable, but we shall not consider that further in this paper.

It remains to comment that to get Theorem A from Theorem B we have to show that given such a set S it is possible to find a Kähler potential whose Hele-Shaw flow develops tangency along S . To do this we first choose Ω_T as in Definition 1.1, and we aim to find a Kähler potential for which we can understand the Hele-Shaw flow backwards for a small time, say for $t \in [T - \epsilon, T]$. As is well known in the Hele-Shaw literature it is not normally the case that the strong Hele-Shaw flow exists backwards in time starting at some Ω_T (for instance if ω is analytic then a necessary condition is that Ω_T has analytic boundary). However, using a previous result of the authors [18], we shall see that this assumption is not necessary as long as one allows a (smooth) modification of the area form near Ω_T (said another way, we make a smooth modification of the permeability that governs the flow). We may then shrink $\Omega_{T-\epsilon}$ down to 0, and expand Ω_T out to ∞ , so as to obtain a flow of sets $\{\Omega_t\}_{t \in (0,1)}$ with properties that ensure that it is the Hele-Shaw flow for some Kähler potential that can be constructed from the flow. Details can be found in Section 3.

Families of Geodesic Rays. As already mentioned, the weak solution $\tilde{\Phi}$ to the Dirichlet problem for the HMAE over the punctured disc $\overline{\mathbb{D}}^\times$ with S^1 -invariant boundary data is by definition a weak geodesic ray in the space of positive potentials on X . If this solution is

regular (by which we mean it is smooth and strictly ω -plurisubharmonic along the fibres over \mathbb{D}^\times) then it gives a genuine geodesic in this space, i.e. a smooth geodesic in the space of Kähler metrics. For this reason regularity of the weak geodesic ray is of interest, and following [19] we know that on \mathbb{P}^1 and \mathbb{D} , this regularity is intimately related to the topology of the Hele-Shaw flow. To state our theorems in the simplest way, let $B(t)$ denote the geodesic ball in X centred at 0 with area t taken with respect to the metric ω .

Definition. Let $a \in [0, V]$. We say that a collection of subsets $\{\Omega_t\}_{t \in (0, V)}$ of X is *standard as t tends to a* if there exist $\epsilon > 0$ such that

$$\Omega_t = B(t) \text{ for } |t - a| < \epsilon.$$

Theorem C. Let $X = \mathbb{P}^1$ or $\mathbb{X} = \mathbb{D}$ and suppose the flow Hele-Shaw $\{\Omega_t\}_{t \in (0, V)}$ for a Kähler form $\omega + dd^c \phi$ satisfies

- (1) $\{\Omega_t\}_{t \in (0, V)}$ is smoothly bounded and varies smoothly with non-vanishing normal velocity,
- (2) Ω_t is simply connected for all $t \in (0, V)$,
- (3) if $X = \mathbb{P}^1$ then $\{\Omega_t\}_{t \in (0, V)}$ is standard as t tends to 1.

Then the weak geodesic ray obtained as the Legendre transform of the Hele-Shaw envelopes $\{\psi_t\}$ is regular, and so defines a smooth geodesic ray in the space of Kähler metrics on X .

Of course, for this theorem to have any content we must be able to provide examples of potentials ϕ for which the Hele-Shaw has these properties. An interesting case of this is given by a result of Hedenmalm-Shimorin:

Theorem HS. (Hedenmalm-Shimorin [10]) Let $(X, \omega) = (\mathbb{D}, \omega_P)$ and suppose that ϕ is taken so that the Kähler form $\omega_P + dd^c \phi$ is analytic and hyperbolic. Then the Hele-Shaw flow $\{\Omega_t\}$ for $\omega + dd^c \phi$ is smoothly bounded, smoothly varying, and simply connected for all $t \in (0, \infty)$.

Another class of examples can be constructed from an observation due to Berndtsson (following a question of Zelditch) which says that *any* reasonable smooth increasing family of simply connected domains is the Hele-Shaw flow for some smooth Kähler potential, see Theorem 3.1 (we observe that what we then get for \mathbb{P}^1 turns out to be essentially the same geodesic ray as that described by Donaldson [6, p24]).

Of course it is trivial to construct families of domains $\{\Omega_t\}$ that satisfy the required hypotheses, and thus combining with Theorem C gives an easy way to construct explicit families of smooth geodesic rays in the space of Kähler metrics on \mathbb{P}^1 (resp. on \mathbb{D}). In particular we have that any hyperbolic analytic Kähler metric on \mathbb{D} is the starting point for some canonical smooth geodesic ray.

Acknowledgements. We wish to thank the Simons Center for Geometry and Physics for inviting the authors to the “Large N Program” in the Spring of 2015, in particular Steve Zelditch for his role in organising this program as well as his interest in our previous work which led directly to the material in this note. We also thank the other participants, in particular Bo Berndtsson for his interest and assistance.

During this work JR was supported by an EPSRC Career Acceleration Fellowship (EP/J002062/1). DWN has received funding from the People Programme (Marie Curie Actions) of the European Union’s Seventh Framework Programme (FP7/2007-2013) under REA grant agreement no 329070

2. THE HELE SHAW FLOW

2.1. Definition and Preliminaries. Suppose (X, ω) is a one-dimensional simply-connected Kähler manifold and $\phi \in C^\infty(X) \cap L^\infty(X)$ is such that $\omega_\phi := \omega + dd^c \phi$ is Kähler. Let $V := \int_X \omega \in (0, \infty]$ and fix an origin $0 \in X$. We use the convention $d^c = \frac{1}{2\pi}(\bar{\partial} - \partial)$ so $dd^c \log |z|^2 = \delta_0$. On \mathbb{P}^1 we always having in mind a chart $0 \in \mathbb{C} \subset \mathbb{P}^1$ with coordinate z so the Fubini-Study metric ω_{FS} has local potential $\log(1 + |z|^2)$ on \mathbb{C} giving \mathbb{P}^1 area 1. We let dA denote the Lebesgue measure on \mathbb{C} .

Definition 2.1. For $t \in (0, V)$ set

$$\psi_t := \sup\{\psi : X \rightarrow \mathbb{R} \cup \{-\infty\} : \psi \text{ is usc and } \psi \leq \phi \text{ and } \omega + dd^c \psi \geq 0 \text{ and } \nu_0(\psi) \geq t\}.$$

Here ν_0 denotes the Lelong number at 0, so $\nu_0(\psi) \geq t$ means that $\psi(z) \leq t \ln |z|^2 + O(1)$ near 0. As the upper semi-continuous regularisation of ψ_t is itself a candidate for the envelope defining ψ_t , we see that ψ_t is usc.

Definition 2.2. For $t \in (0, V)$ set

$$\Omega_t := \{z \in X : \psi_t(z) < \phi(z)\}. \quad (2)$$

It is easy to see that if ϕ is replaced by $\phi + h$ for some harmonic function h then ψ_t is replaced by $\psi_t + h$. Thus Ω_t depends only on ω_ϕ .

Definition 2.3. (Hele-Shaw flow) We refer to collection of sets $\{\Omega_t\}_{t \in (0, V)}$ as the *Hele-Shaw flow* associated to (X, ω_ϕ) and the collection $\{\psi_t\}_{t \in (0, V)}$ as the *Hele-Shaw envelopes* associated to (X, ω, ϕ) .

Remark 2.4. What we have called the Hele-Shaw flow is often called the “weak Hele-Shaw flow”. If $(a, b) \subset (0, V)$ we will also refer to the subcollection $\{\Omega_t\}_{t \in (a, b)}$ as the Hele-Shaw flow and similarly for the envelopes.

We first record some basic properties of this flow:

Proposition 2.5.

- (1) Ω_t is an open connected set containing the origin for all $t \in (0, V)$.
- (2) $\partial\Omega_t$ has measure zero.
- (3) ψ_t is $C^{1,1}$ on $X \setminus \{0\}$.
- (4)

$$\omega_{\psi_t} = (1 - \chi_{\Omega_t})\omega_\phi + t\delta_0$$

in the sense of currents. Here χ_A denotes the characteristic function of a set A , and δ_0 the Dirac delta.

- (5) For $t \in (0, V)$ we have

$$\int_{\Omega_t} \omega_\phi = t.$$

Proof. This is standard material for the Hele-Shaw flow, and the details are given in [19, Proposition 1.1] (the cited reference is for $X = \mathbb{P}^1$, but the same proof applies for \mathbb{D} or \mathbb{C}). By uniformization, X is conformally equivalent to one of these three, which is enough to prove the statement in general. \square

Our next Lemma says that the Hele-Shaw flow is local, by which we mean Ω_t depends only on the restriction of ω_ϕ to a neighbourhood of $\bar{\Omega}_t$.

Lemma 2.6 (Locality of the Hele-Shaw Flow). *Let $X' \subset X$ be an open simply-connected set containing 0. Suppose ω_ϕ and $\omega_{\tilde{\phi}}$ are Kähler forms on X such that $\phi|_{X'} = \tilde{\phi}|_{X'}$ and denote the Hele-Shaw flow for (X, ω_ϕ) and $(X, \omega_{\tilde{\phi}})$ by Ω_t and $\tilde{\Omega}_t$ respectively (and similarly for ψ_t and $\tilde{\psi}_t$).*

Then

$$\Omega_t = \tilde{\Omega}_t \text{ and } \psi_t = \tilde{\psi}_t \text{ as long as } \Omega_t \Subset X'.$$

Proof. Define γ to be equal to ψ_t on X' and equal to $\tilde{\phi}$ on $X \setminus X'$. Since Ω_t is relatively compact in X' we see that $\gamma = \phi = \tilde{\phi}$ on $X' \setminus \Omega_t$. Clearly $\gamma \leq \phi$ on X' , so as $\phi|_{X'} = \tilde{\phi}|_{X'}$ we have γ is bounded above by $\tilde{\phi}$ and clearly $\omega + dd^c \gamma \geq 0$.

Hence γ is a candidate for the envelope defining $\tilde{\psi}_t$ giving $\gamma \leq \tilde{\psi}_t$. Thus

$$\psi_t|_{X'} \leq \tilde{\psi}_t|_{X'}.$$

This implies $X' \setminus \Omega_t \subset X' \setminus \Omega'_t$. So as Ω'_t is connected by Proposition 2.5 and intersects Ω_t (they both contain the origin) we must actually have $\Omega'_t \subset \Omega_t$. In particular, Ω'_t is relatively compact in X' , so we may run the argument with Ω_t and Ω'_t swapped to conclude $\Omega_t \subset \Omega'_t$. \square

2.2. The Strong Hele-Shaw flow. We shall also need the notion of a strong solution to the Hele-Shaw flow. We shall only consider this in the plane, so suppose $\{\Omega_t\}_{t \in (a,b)}$ is a smooth increasing family of domains of \mathbb{C} . By this we mean each Ω_t is smoothly bounded and varies smoothly, so locally $\partial\Omega_t$ is the graph of a smooth function that varies smoothly with t . So if n denotes the outward unit normal vector field n on $\partial\Omega_{t_0}$ for some t_0 , then for t close to t_0 we can write $\partial\Omega_t = \{x + f(x, t)n_x : x \in \partial\Omega_{t_0}\}$ for some smooth function $f_t(x) = f(x, t)$ on $\partial\Omega_{t_0}$ that is positive for $t > t_0$ and negative for $t < t_0$. Then the *normal velocity* of $\partial\Omega_{t_0}$ is defined to be

$$V_{t_0} := \left. \frac{df_t}{dt} \right|_{t=0} n.$$

Now assume also each Ω_t contains the origin. For each t let $p_t(z) := -G_{\Omega_t}(z, 0)$ where G_{Ω_t} denotes the Green's function for Ω_t with logarithmic singularity at the origin. Thus

$$p_t = 0 \text{ on } \partial\Omega_t \text{ and } \Delta p_t = -\delta_0.$$

The statement that p_t exists and is smooth on $\overline{\Omega_t} \setminus \{0\}$ is classical (this follows immediately from regularity of the Dirichlet problem for the Laplacian (e.g. [11, Proposition 1.3.11]) which can be found, for instance, in [9, Chapter 6]). We also fix a smooth area form η on \mathbb{C} which we write as

$$\eta = \frac{1}{\kappa} dA$$

where dA is the Lebesgue measure and κ is a strictly positive real-valued smooth function on \mathbb{C} .

Definition 2.7. (Strong Hele-Shaw flow) We say that $\{\Omega_t\}_{t \in (a,b)}$ is the *strong Hele-Shaw flow* if

$$V_t = -\kappa \nabla p_t \text{ on } \partial\Omega_t \text{ for } t \in (a, b) \quad (3)$$

where V_t is the normal velocity of $\partial\Omega_t$.

Remark 2.8. The strong Hele-Shaw flow has an interpretation as the flow of a fluid moving between two plates in a medium which has a permeability encoded by the function κ , under injection of fluid at the origin (see [18] for a discussion, and also [8] for a comprehensive account of the subject, which for the most part considers the case where $\kappa \equiv 1$). When

necessary to emphasise the dependence on the area form we refer to this as the strong Hele-Shaw flow with respect to η (or with respect to κ).

We shall now prove that a strong Hele-Shaw flow of simply connected domains is also the Hele-Shaw flow defined using envelopes, as in Definition 2.3.

Lemma 2.9. (Richardson [16]) *Suppose that $\{\Omega_t\}_{t \in (a,b)}$ is a smooth family of strictly increasing simply connected domains in \mathbb{C} containing the origin that satisfies*

$$V_t = -\kappa \nabla p_t \text{ on } \partial\Omega_t \quad (4)$$

as in (3). Then for any integrable subharmonic function h on Ω_t , and $t_0 < t$ we have

$$\int_{\Omega_t \setminus \Omega_{t_0}} h \frac{dA}{\kappa} \geq (t - t_0)h(0).$$

Proof. We compute

$$\begin{aligned} \frac{d}{dt} \int_{\Omega_t} h \frac{1}{\kappa} dA &= \int_{\partial\Omega_t} h \frac{V_t}{\kappa} ds = - \int_{\partial\Omega_t} h \frac{\partial p_t}{\partial n} ds \\ &= \int_{\Omega_t} (p_t \Delta(h) - h(\Delta p_t)) dA - \int_{\partial\Omega_t} p_t \frac{\partial h}{\partial n} ds \geq h(0) \end{aligned}$$

since $\Delta h \geq 0$ and $p_t = 0$ on $\partial\Omega_t$ and $\Delta p_t = -\delta_0$. \square

Corollary 2.10. *With the assumption of the above lemma, suppose that $a = 0$ and Ω_t tends to $\{0\}$ as $t \rightarrow 0$, that is given any neighbourhood U of the origin $\Omega_t \subset U$ for t sufficiently small. Then for any integrable subharmonic function h on Ω_t , we have*

$$\int_{\Omega_t} h \frac{dA}{\kappa} \geq th(0).$$

and moreover equality holds if h is holomorphic.

Proof. Taking the limit as $t_0 \rightarrow 0$ in the above Lemma gives the first statement. The second follows as if h is holomorphic then h and $-h$ are subharmonic. \square

Proposition 2.11. (Gustafsson) *Suppose that $\{\Omega_t\}_{t \in (0,b)}$ is a smooth family of strictly increasing simply connected domains that is the strong Hele-Shaw flow with respect to κ , and assume $\{\Omega_t\}_{t \in (0,b)}$ tends to $\{0\}$ as $t \rightarrow 0$. Set*

$$\phi(z) = \int_{\mathbb{C}} \log |z - \zeta|^2 \frac{dA_\zeta}{\kappa(\zeta)} - \log(1 + |z|^2) \text{ for } z \in \mathbb{C}.$$

Then $\{\Omega_t\}_{t \in (0,b)}$ is the Hele-Shaw flow with respect to $\omega_\phi := dd^c(\log(1 + |z|^2) + \phi)$.

Proof. For the proof we shall write $\Omega_t^w := \{z \in X : \psi_t(z) < \phi(z)\}$ for the Hele-Shaw flow with respect to ω_ϕ , so the goal is to prove $\Omega_t^w = \Omega_t$. Define

$$\tilde{\psi}_t(z) := \int_{\Omega_t^c} \log |z - \zeta|^2 \frac{dA_\zeta}{\kappa(\zeta)} - \log(1 + |z|^2) + t \ln |z|^2.$$

Then by construction $\omega_{\tilde{\psi}_t} \geq 0$ and $\nu_0(\tilde{\psi}_t) = t$. As $h(\zeta) := \log |z - \zeta|^2$ is subharmonic and integrable, we get from the previous Corollary that for all $z \in \mathbb{C}$

$$\phi(z) - \tilde{\psi}_t(z) = \int_{\Omega_t} \log |z - \zeta|^2 \frac{dA_\zeta}{\kappa(\zeta)} - t \ln |z|^2 \geq 0. \quad (5)$$

Hence $\tilde{\psi}_t \leq \phi$ making it a candidate for the envelope defining ψ_t , and hence $\tilde{\psi}_t \leq \psi_t$. In fact more is true, and if $z \notin \Omega_t$ then h is holomorphic on Ω_t so equality holds in (5), and hence

$$\tilde{\psi}_t = \psi_t = \phi \text{ on } \Omega_t^c.$$

Now $\tilde{\psi}_t + \log(1+|z|^2)$ and $\psi_t + \log(1+|z|^2)$ are both harmonic on $\Omega_t \setminus \{0\}$ (by Proposition 2.5(4)) with Lelong number one at 0. Hence by the maximum principle $\tilde{\psi}_t = \psi_t$ on $\Omega_t \setminus \{0\}$ as well. Thus we conclude $\Omega_t^w = \Omega_t$ as desired. \square

Remark 2.12. Although we will not really need it, we remark that there is a converse to this, which says that if the Hele-Shaw domain Ω_T (with respect to $\omega + dd^c \phi$) is a smoothly bounded Jordan domain for some $T \in (0, V)$ then there is an $\epsilon > 0$ such that $\{\Omega_t\}_{t \in (T-\epsilon, T+\epsilon)}$ is actually the strong Hele-Shaw flow. Thus the hypothesis that $\{\Omega_t\}$ varies smoothly in Theorem C (as well in Definition 1.1) is redundant. The proof of this statement follows easily from the work in [18]; specifically from [18, Remark 3.12] the Hele-Shaw domains Ω_t all lift to holomorphic curves Σ_T in $\mathbb{C} \times \mathbb{P}^1$ with boundary contained in the submanifold given as the graph of $\frac{\partial \phi}{\partial \bar{z}}$. The hypothesis on Ω_T imply that Σ_T is a holomorphic disc, at which point we can run the proof of [18, Theorem 2.2].

Finally, we state two previous results of the authors that give existence results for the strong Hele-Shaw flow. The first says this flow always exists for small time.

Theorem 2.13. [18, Theorem 2.1] *The Hele-Shaw flow for any Kähler form $\omega + dd^c \phi$ is the strong Hele-Shaw flow for short time $t \in (0, \epsilon)$, $\epsilon > 0$, and in this range is diffeomorphic to the standard flow $B(t)$.*

The second says that any simply connected bounded Jordan domain Ω is part of a strong Hele-Shaw flow, both backwards and forwards in time, as long as one allows a modification of the area form inside Ω .

Theorem 2.14. [18, Theorem 2.2, Remark 7.1] *Let η be a smooth area form on \mathbb{C} and Ω a smoothly bounded Jordan domain containing the origin. Then there exists a smooth area form η' on a neighbourhood of $\overline{\Omega}^c$ such that $\eta = \eta'$ on Ω^c and so that $\Omega = \Omega_T$ is part of a strong Hele-Shaw flow $\{\Omega_t\}_{t \in (T-\epsilon, T+\epsilon)}$ with respect to η' .*

3. DESIGNER POTENTIALS

In this section we show how to produce potentials with particular prescribed properties (we do this only on \mathbb{P}^1 but a similar story holds for \mathbb{D}). We first show that any (reasonable) strictly increasing family of smooth domains in \mathbb{P}^1 is the Hele-Shaw flow for some smooth Kähler potential. Recall $B(t)$ denotes the geodesic ball centred at the origin of area t taken with respect to ω_{FS} .

Theorem 3.1. *Suppose $\{\Omega_t\}_{t \in (0,1)}$ is a family of subsets of \mathbb{P}^1 that is*

- (1) *smoothly bounded, varies smoothly, and is simply connected for all t ,*
- (2) *increasing, i.e. $\Omega_t \subseteq \Omega_{t'}$ for $t < t'$, with non-vanishing normal velocity of the boundary $\partial\Omega_t$, and*
- (3) *standard as t tends to 0 and as t tends to 1.*

Then there exists a smooth $\phi \in C^\infty(\mathbb{P}^1)$ such that $\{\Omega_t\}_{t \in (0,1)}$ is the Hele-Shaw flow with respect to the Kähler form $\omega_{FS} + dd^c \phi$.

Proof. The idea of the proof is to construct a smooth function κ on \mathbb{C} such that $\{\Omega_t\}$ is the strong Hele-Shaw flow with respect to the permeability κ . Since $\{\Omega_t\}_{t \in (0,1)}$ is assumed to be standard as t tends to 1 we have $\Omega_t \subset \mathbb{C}$ for all $t \in (0,1)$ and so by Lemma 2.6 we may as well consider the Hele-Shaw flow as taking place in \mathbb{C} . Let p_t satisfy

$$p_t = 0 \text{ on } \partial\Omega_t \text{ and } \Delta p_t = -\delta_0.$$

As already mentioned, the fact that p_t exists and is smooth on $\overline{\Omega_t} \setminus \{0\}$ is classical. What is also true is that p_t varies smoothly with t ; this is presumably also well-known in some circles, but since we were not able to find a convenient reference we give a proof in Appendix A.

Assuming this smoothness for now, we use p_t to define a function κ by requiring that

$$V_t = -\kappa \nabla p_t \text{ on } \partial\Omega_t \text{ for } t \in (0,1) \quad (6)$$

where V_t is the normal velocity of $\partial\Omega_t$. Since $\{\Omega_t\}_{t \in (0,1)}$ is increasing smoothly and V_t was assumed to be non-vanishing we see that κ is a well-defined strictly positive smooth function on $\mathbb{C} \setminus \{0\}$.

Now we use the assumption that $\{\Omega_t\}_{t \in (0,1)}$ is standard as t tends to zero to deduce that κ extends to a smooth function over 0. Assume $t \ll 1$. By explicit calculation with the Fubini-Study metric one computes that $\Omega_t = \{z \in \mathbb{C} : |z| < R_t\}$ where

$$R_t := \left(\frac{t}{1-t} \right)^{1/2}.$$

Thus $p_t(z) = -\frac{1}{4\pi}(\log |z|^2 - \log(R_t^2))$. Clearly κ is radially symmetric near 0, so it is sufficient to compute it at a point $z_t := (R_t, 0)$ for small t . To do so observe that at z_t we have

$$\nabla p_t = -\frac{1}{2\pi R_t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

On the other hand the normal velocity of $\partial\Omega_t$ at the point z_t is $\frac{dR_t}{dt} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and so the defining equation (6) for κ becomes

$$\frac{1}{2R_t} \frac{1}{(1-t)^2} = \frac{\kappa(z_t)}{2\pi R_t}.$$

After some calculation this yields

$$\kappa(z) = \pi(1 + |z|^2)^2 \text{ near } z = 0 \quad (7)$$

which clearly extends smoothly over $z = 0$.

Now define

$$\phi(z) = \int_{\mathbb{C}} \log |z - \zeta|^2 \frac{dA_\zeta}{\kappa(\zeta)} - \log(1 + |z|^2) \text{ for } z \in \mathbb{C} \quad (8)$$

which is a smooth function on \mathbb{C} chosen so that

$$dd^c(\log(1 + |z|^2) + \phi) = \frac{dA}{\kappa} \text{ on } \mathbb{C}. \quad (9)$$

Using that $\{\Omega_t\}_{t \in (0,1)}$ is standard as t tends to infinity we have that (7) also holds for $|z|$ sufficiently large. We claim this implies ϕ extends to a smooth function on \mathbb{P}^1 and ω_ϕ is Kähler on \mathbb{P}^1 . To see this, start with the identity

$$\frac{1}{\pi} \int_{\mathbb{C}} \frac{\log |z - \zeta|^2}{(1 + |\zeta|^2)^2} dA_\zeta = \log(1 + |z|^2)$$

(this can be seen by noting that the difference is harmonic on \mathbb{C} bounded and equal to zero at $z = 0$). Now the same calculation as above means the assumption that $\{\Omega_t\}$ is standard as t tends to 1 implies $C > 0$ such that (7) holds on $\{|z| > C\}$. Therefore

$$\phi(z) = \frac{1}{\pi} \int_{|z| < C} \log |z - \zeta|^2 \left(\frac{\pi}{\kappa(\zeta)} - \frac{1}{(1 + |\zeta|^2)^2} \right) dA_\zeta$$

which one sees extends smoothly over $z = \infty$ in such a way that makes ω_ϕ Kähler as claimed. (We remark that this can also be seen abstractly, since the flow being standard near 0 and ∞ means that κ has to agree with the permeability for the standard flow for $(\mathbb{P}^1, \omega_{FS})$.)

Now by construction $\{\Omega_t\}$ is the strong Hele-Shaw flow with respect to κ , and hence by Proposition 2.11, is the strong Hele-Shaw flow for $\omega_{FS} + dd^c \phi$ as desired. \square

Remark 3.2. We observe that the above proof actually shows slightly more, namely that if $\{\Omega_t\}_{t \in (0, T]}$ is a smooth family of strictly increasing domains that is standard as $t \rightarrow 0$ then setting $X' := \Omega_T$ there exists a $\phi \in C^\infty(X')$ such that $\{\Omega_t\}_{t \in (0, T)}$ is the Hele-Shaw flow for (X', ω_ϕ) .

Now let S be a finite union of points and non-intersecting smooth embedded curve segments in $\mathbb{P}^1 \setminus \{0\}$. Using similar ideas to above we now show that there are Kähler potentials whose Hele-Shaw flow is smoothly bounded and simply connected until it develops a tangency along S .

Proposition 3.3. *There exists a $\phi \in C^\infty(\mathbb{P}^1)$ such that ω_ϕ is Kähler and whose associated Hele-Shaw flow develops tangency along S .*

Proof. It is clear that one can find a simply connected domain Ω containing 0 such that $\partial\Omega$ is the image of a smooth locally embedded curve γ intersecting itself tangentially precisely along S and so $\overline{\Omega}_t \setminus S$ is connected as in Figure 1 (use induction on the number of components of S). Let

$$T := \int_{\Omega} \omega_{FS}.$$

We construct the Hele-Shaw flow backwards starting at $\Omega_T := \Omega$.

Pick a point z_i in each connected component of $\mathbb{P}^1 \setminus \overline{\Omega}_T$, and let π be the projection from the universal cover Σ of \mathbb{P}^1 with the points z_i removed. Then γ lifts to a smooth embedded curve in Σ and so $\pi^{-1}(\Omega_T)$ is a disjoint union of copies of Ω_T . We pick one of them and call it Ω' which is smoothly bounded and simply connected. Then Theorem 2.14 implies that there exists a smooth area form η' on a neighbourhood of $\Sigma \setminus \Omega'$, equal to $\eta := \pi^* \omega_{FS}$ on $\Sigma \setminus \Omega'$, such that the strong Hele-Shaw flow exists starting from Ω' with respect to η' for a short while backwards in time. We denote the projection of this Hele-Shaw flow to \mathbb{P}^1 by $\{\Omega_t\}_{t \in (T-\epsilon, T]}$.

We then extend this to a family of domains Ω_t , $t \in (0, T)$, in \mathbb{P}^1 , with the properties as in Theorem 3.1, so by this Theorem and Remark 3.2 we have an area form ω' on Ω_T such that $\{\Omega_t\}_{t \in (0, T)}$ is a strong Hele-Shaw flow with respect to ω' . We also have that $\omega' = \omega_{FS}$ on $\Omega_T \setminus \Omega_{T-\epsilon}$. We can thus extend ω' to a smooth Kähler form on \mathbb{P}^1 by letting it be equal to ω_{FS} on $\mathbb{P}^1 \setminus \Omega_T$. Thus $\{\Omega_t\}_{t \in (0, T)}$ is the strong Hele-Shaw flow with respect to the area form ω' on \mathbb{P}^1 , and thus also the Hele-Shaw flow by Proposition 2.11.

On the other hand, by the continuity of the Hele-Shaw flow (applied on Σ) it follows that Ω_T is the Hele-Shaw domain of ω' at time T . Thus if ϕ is a smooth function so that $\omega' = \omega_{FS} + dd^c \phi$ we get that the Hele-Shaw flow with respect to ϕ develops a tangency along S at time T . \square

Remark 3.4. If we assume in addition that S is such that one can find such an Ω_T with real-analytic boundary, then instead of the previous result of the authors (Theorem 2.14) one can use the classical short-time existence result of the Hele-Shaw backwards starting with simply connected domain with real analytic boundary.

4. DIRICHLET PROBLEM FOR THE HOMOGENEOUS MONGE-AMPÈRE EQUATION

4.1. Preliminary definitions. We first recall two versions of the Dirichlet Problem for the complex homogeneous Monge-Ampère Equation, first over the disc and second over the punctured disc. In the following (X, ω) will be either \mathbb{P}^1 with the Fubini-Study metric ω_{FS} with area 1, or X is the unit disc \mathbb{D} with the Poincarè metric ω_P . Again we let $\phi \in C^\infty(X)$ be such that $\omega + dd^c \phi$ is Kähler and $\pi_X : X \times \mathbb{D} \rightarrow X$ and $\pi_{\mathbb{D}} : X \times \mathbb{D} \rightarrow \mathbb{D}$ be the projections.

Definition 4.1. (Weak Solution)

(1) Let

$$\Phi := \sup \left\{ \begin{array}{l} \psi : X \times \overline{\mathbb{D}} \rightarrow \mathbb{R} \cup \{-\infty\} : \psi \text{ is usc, } \pi_X^* \omega + dd^c \psi \geq 0 \\ \text{and } \psi(z, \tau) \leq \phi(\tau z) \text{ for } (z, \tau) \in X \times \partial \mathbb{D}. \end{array} \right\}. \quad (10)$$

(2) Let

$$\tilde{\Phi} := \sup \left\{ \begin{array}{l} \psi : X \times \overline{\mathbb{D}} \rightarrow \mathbb{R} \cup \{-\infty\} : \psi \text{ is usc, } \pi_X^* \omega + dd^c \psi \geq 0 \\ \text{and } \psi(z, \tau) \leq \phi(z) \text{ for } (z, \tau) \in X \times \partial \mathbb{D} \text{ and } \nu_{(0,0)}(\psi) \geq 1 \end{array} \right\}. \quad (11)$$

So the difference between these two definitions is that in the second the boundary data is S^1 -invariant but has an additional requirement of giving a prescribed singularity at the point $(0, 0)$. However these two quantities carry the same information as given by:

Proposition 4.2. *We have that*

$$\Phi(z, \tau) + \ln |\tau|^2 + \ln(1 + |z|^2) = \tilde{\Phi}(\tau z, \tau) + \ln(1 + |\tau z|^2) \text{ for } (z, \tau) \in \mathbb{P}^1 \times \overline{\mathbb{D}}^\times.$$

Proof. This is proved in [19, Proposition 2.3] using a blowup, but it can also be seen directly from the definition that $\Phi(z, \tau) + \ln |\tau|^2 + \ln(1 + |z|^2) - \ln(1 + |\tau z|^2)$ is a candidate for the envelope defining $\tilde{\Phi}(\tau z, \tau)$, giving one inequality and the other inequality is proved similarly. \square

Definition 4.3. (Regular solution) We say that Φ is *regular* on an open subset $S \subset X \times \overline{\mathbb{D}}$ if it is smooth on S and the restriction of $\pi_X^* \omega + dd^c \Phi$ to $S_\tau := \pi_{\mathbb{D}}^{-1}(\tau) \cap S$ is strictly positive for all $\tau \in \mathbb{D}$. Similarly we say $\tilde{\Phi}$ is *regular* on S if it is smooth on $S \setminus \{0\}$ and the restriction of $\pi_X^* \omega + dd^c \tilde{\Phi}$ to S_τ is strictly positive for all $\tau \in \overline{\mathbb{D}}^\times$.

Finally we say that Φ (resp. $\tilde{\Phi}$) is *regular* if it is regular on all of $X \times \overline{\mathbb{D}}$ (resp. $X \times \overline{\mathbb{D}}^\times$).

By well-known arguments, $\tilde{\Phi}$ is usc, $\pi_X^* \omega + dd^c \tilde{\Phi} \geq 0$ and $(\pi_X^* \omega + dd^c \tilde{\Phi})^2 = 0$ away from $(0, 0)$ and $\tilde{\Phi}(z, \tau) = \phi(z)$ for $\tau \in \partial \mathbb{D}$. Moreover it is not hard to show that $\tilde{\Phi}$ is locally bounded away from $(0, 0)$ and $\nu_{(0,0)} \tilde{\Phi} = 1$. Thus $\tilde{\Phi}$ is the weak solution to Dirichlet problem to the Homogeneous Monge-Ampère Equation with boundary data consisting of $\phi(z)$ on $X \times \partial \mathbb{D}$, and this prescribed singularity at $(0, 0)$. Thinking of $s := -\ln |\tau|^2$ for $\tau \in \mathbb{D}^\times$ as a time variable let $\tilde{\Phi}|_s(\cdot) = \tilde{\Phi}(\cdot, s)$. Then the map

$$s \mapsto \omega + dd^c \tilde{\Phi}_s$$

is a weak geodesic ray in the space of weak Kähler metrics that starts with $\omega + dd^c \phi$ and has limit the singular potential $\omega + dd^c \ln |z|^2$ as s tends to infinity. Moreover if $\tilde{\Phi}$ is regular this is a smooth geodesic in the space of Kähler metrics.

Similarly Φ is the weak solution to the same Dirichlet problem over $X \times \overline{\mathbb{D}}$ with prescribed boundary $\phi(\tau z)$ over $X \times \partial\mathbb{D}$.

4.2. The Duality Theorem. The duality between $\tilde{\Phi}$ and the Hele-Shaw envelopes ψ_t is provided by the following:

Theorem 4.4. (*Ross-Witt-Nyström [19, Theorem 2.7]*) *Let ψ_t be the Hele-Shaw envelopes associated to (X, ω, ϕ) and $\tilde{\Phi}$ be the weak solution to the Homogeneous Monge-Ampère Equation as defined in (11). Then*

$$\psi_t(z) = \inf_{|\tau|>0} \{ \tilde{\Phi}(z, \tau) - (1-t) \ln |\tau|^2 \} \quad (12)$$

and

$$\tilde{\Phi}(z, \tau) = \sup_t \{ \psi_t(z) + (1-t) \ln |\tau|^2 \}. \quad (13)$$

Remark 4.5. In [19] this theorem is proved when $X = \mathbb{P}^1$, but precisely the same case works when X is the disc.

5. REGULARITY OF GEODESIC RAYS

Let (X, ω) be either $(\mathbb{P}^1, \omega_{FS})$ or (\mathbb{D}, ω_P) with $V := \int_X \omega \in \{1, \infty\}$, and let $\phi \in C^\infty(X)$ be such that $\omega + dd^c \phi$ is Kähler (when $X = \mathbb{D}$ we also assume ϕ to be bounded). We continue with the notation from the previous section, so $\tilde{\Phi}$ is as defined in (11).

Definition. Let $f: \mathbb{D} \rightarrow X$ be holomorphic. We say that the graph of f is a *harmonic disc* for $\tilde{\Phi}$ if $\tilde{\Phi}$ is $\pi_X^* \omega$ -harmonic along the graph of $f|_{\mathbb{D}^\times}$. That is, the restriction of $\pi_X^* \omega + dd^c \tilde{\Phi}$ to $\{(f(\tau), \tau) : \tau \in \mathbb{D}, \tau \neq 0\}$ vanishes.

Definition 5.1. We define

$$H(z, \tau) := \frac{\partial}{\partial s} \tilde{\Phi}(z, e^{-s/2}) \text{ for } (z, \tau) \in X \times \overline{\mathbb{D}}^\times$$

where $s := -\ln |\tau|^2$ (so when $|\tau| = 1$ and thus $s = 0$ we take the right derivative).

We recall that by a result of Chen [5], with complements by Błocki [4], the function $\tilde{\Phi}$ is $C^{1,1}$ and thus this definition makes sense, and H is continuous (even Lipschitz but we will not use this).

Lemma 5.2. *The function H is constant along any harmonic disc of $\tilde{\Phi}$.*

Proof. Let $\Omega = \pi_X^* \omega_{FS} + dd^c \tilde{\Phi}$. Then one calculates that $dH = \iota_\zeta \Omega$ where ζ is the infinitesimal action of the natural S^1 -action given by $e^{i\theta} \cdot (z, \tau) = (e^{i\theta} z, e^{-i\theta} \tau)$ for $(z, \tau) \in X \times \mathbb{D}^\times$ (this is essentially as in [17, Theorem 3.14]). This proves that H is constant along any harmonic leaf as required. \square

The connection with the Hele-Shaw flow is given by:

Proposition 5.3.

$$H(z, 1) + 1 = \sup\{t : \psi_t(z) = \phi(z)\} = \sup\{t : z \notin \Omega_t\}.$$

Proof. This is [19, Proposition 2.8] and for convenience we repeat the proof here. From (13) if $\psi_t(z) = \phi(z)$ then

$$\tilde{\Phi}(z, e^{-s/2}) \geq (t-1)s + \phi(z)$$

and thus

$$H(z, 1) \geq \sup\{t : \psi_t(z) = \phi(z)\} - 1.$$

Suppose $\psi_t(z) \leq \phi(z) + a$ for some $a < 0$. One can easily check that for a fixed z the function $t' \mapsto \psi_{t'}(z)$ is concave and decreasing in t' , so for $t \leq t' < V$ and $s \geq 0$ we have $\psi_{t'}(z) + (t' - 1)s \leq \phi(z) + a$. On the other hand we always have $\psi_{t'} \leq \phi$ so if $0 \leq t' \leq t$ then $\psi_{t'}(z) + (t' - 1)s \leq \phi(z) + (t - 1)s$. Putting this together with (13) gives

$$\tilde{\Phi}(z, e^{-s/2}) \leq \phi(z) + \max((t - 1)s, a)$$

and so $H(z, 1) \leq t - 1$, which proves the proposition. \square

Theorem C. Suppose the flow Hele-Shaw $\{\Omega_t\}_{t \in (0, V)}$ for a Kähler form $\omega + dd^c \phi$ satisfies

- (1) $\{\Omega_t\}_{t \in (0, V)}$ is smoothly bounded and varies smoothly with non-vanishing normal velocity,
- (2) Ω_t is simply connected for all $t \in (0, V)$,
- (3) if $X = \mathbb{P}^1$ then $\{\Omega_t\}_{t \in (0, V)}$ is standard as t tends to 1.

Then the weak geodesic ray (13) obtained as the Legendre transform of the Hele-Shaw envelopes $\{\psi_t\}$ is regular, and so defines a smooth geodesic ray in the space of Kähler metrics on X .

Proposition 5.4. *With the assumptions as in Theorem C let $f_t : \mathbb{D} \rightarrow \Omega_t$ be a Riemann-mapping for Ω_t that maps 0 to 0. The graph of f_t is a harmonic disc for $\tilde{\Phi}$. Another harmonic disc is given by the graph of the constant function $f_0(\tau) \equiv 0$ and when $X = \mathbb{P}^1$ yet another is given by the graph of $f_\infty(\tau) \equiv \infty$. Finally the union of all these harmonic discs form a smooth foliation of $X \times \overline{\mathbb{D}}^\times$.*

Proof. From (13) we have that

$$\tilde{\Phi}(f(\tau), \tau) \geq \psi_t(f(\tau)) + (1 - t) \ln |\tau|^2.$$

Now the left hand side is $\pi_X^* \omega$ -subharmonic whilst the right hand side is $\pi_X^* \omega$ -harmonic for $\tau \neq 0$. Since $f(\tau)$ escapes to infinity in Ω_t as $|\tau| \rightarrow 1$ the left hand side approaches the right hand side as $|\tau| \rightarrow 1$. Hence by the maximum principle we deduce that in fact

$$\tilde{\Phi}(f(\tau), \tau) = \psi_t(f(\tau)) + (1 - t) \ln |\tau|^2.$$

which is $\pi_X^* \omega$ -harmonic for $\tau \neq 0$. The fact that the graph of f_0 is a harmonic disc is clear as the boundary data is invariant for the S^1 -action $e^{i\theta} \cdot (z, \tau) = (e^{i\theta} z, e^{-i\theta} \tau)$, and hence so is $\tilde{\Phi}$, meaning that $\tilde{\Phi}$ is actually constant along the graph of f_0 (and similarly for f_∞ when $X = \mathbb{P}^1$).

That these harmonic discs $X \times \overline{\mathbb{D}}$ do not intersect for different values of t is clear from Lemma 5.2 and Proposition 5.3 (which together say that $H \equiv t - 1$ on f_t for $t \in (0, V)$) and the union of all these discs cover $X \times \overline{\mathbb{D}}$ since the union of the Ω_t cover X .

Now by Theorem 2.13, the foliation is diffeomorphic to the standard one $B(t)$ for $t \in (0, \epsilon)$ for some $\epsilon > 0$. This proves the foliation by harmonic discs is smooth in a neighbourhood of $\{0\} \times \overline{\mathbb{D}}$. When $X = \mathbb{P}^1$, the assumption that the Hele-Shaw flow is standard as t tends to 1 ensures the foliation is also smooth near $\{\infty\} \times \overline{\mathbb{D}}$. That these harmonic discs give a smooth foliation in the remaining part of $X \times \overline{\mathbb{D}}$ is immediate from the hypothesis that the normal velocity of $\{\Omega_t\}$ is non-vanishing, as we can arrange so that the Riemann mapping $f_t : \mathbb{D} \rightarrow \Omega_t$ that takes 0 to 0 varies smoothly with t since Ω_t varies smoothly in t . \square

Remark 5.5. We remark that the first part of the above is the easy direction of [19, Theorem 3.1] which actually gives a complete characterisation of harmonic discs in terms of the simply connectedness of the Hele-Shaw flow domains.

Remark 5.6. One can avoid using the previous work of the authors (Theorem 2.13) if one assumes in addition that $\{\Omega_t\}$ is standard as t tends to 0.

Proof of Theorem C. This is an almost immediate consequence of the existence of the smoothly varying foliation by harmonic discs provided by Proposition 5.4 (this is essentially contained in [6]). Let $D = \{(f(\tau), \tau)\}$ be such a harmonic disc away from $\{0\} \times \mathbb{D}$ (and away from $\{\infty\} \times \mathbb{D}$ when $X = \mathbb{P}^1$). Then $\tilde{\Phi}(f(\tau), \tau)$ is harmonic along D with Lelong number one. Thus by the mean-value property of harmonic functions, $\tilde{\Phi}|_D$ can be expressed as an integral of $\tilde{\Phi}$ over ∂D . But $\tilde{\Phi} = \phi$ over ∂D (which is smooth) and the foliation varies smoothly, from which we conclude that $\tilde{\Phi}$ must in fact be smooth. Since the foliation is diffeomorphic to the trivial foliation near $\{0\} \times \mathbb{D}$ one sees that in fact $\tilde{\Phi}$ is also smooth near $\{0\} \times \mathbb{D}$ (and similarly for $\{\infty\} \times \mathbb{D}$ when $X = \mathbb{P}^1$). This proves smoothness of $\tilde{\Phi}$ over $X \times \mathbb{D}^\times$.

For the regularity we argue as follows. For $\tau \neq 0$ let $T_\tau: \pi_{\mathbb{D}}^{-1}(1) \rightarrow \pi_{\mathbb{D}}^{-1}(\tau)$ be the flow along the leaves of the above foliation and set $\Omega_\tau := \pi_X^* \omega_{FS} + dd^c \tilde{\Phi}|_{\pi_{\mathbb{D}}^{-1}(\tau)}$. Then by what is now considered a classical calculation (originally due to Semmes [22] and Donaldson [7], see also [17, Proposition 3.4]) we know $T_\tau^* \Omega_\tau = \Omega_1$. But $\Omega_1 = \omega_\phi$ is certainly strictly positive, and hence Ω_τ is strictly positive as well. \square

6. EXPLICIT SINGULARITIES

We now give a proof of Theorem B and show that a potential whose Hele-Shaw flow that develops a tangency along a set S gives a singularity of the associated weak solution.

Example 6.1. The reader may find the following simple example instructive. Suppose ϕ develops tangency at a single point $S = \{z_0\}$. Then we may find smooth coordinates (x, y) centered at z_0 such that

$$\partial\Omega_t = \{y = x^2 + (t_0 - t)\} \cup \{y = -x^2 - (t_0 - t)\} \text{ near } z_0.$$

Let $h := H(\cdot, 1)$ where H is as defined in (5.1). Notice that if $|y|$ is sufficiently small then $(0, \pm y)$ lies in Ω_t for t sufficiently close to t_0 . Thus Proposition 5.3 says that for $|y|$ sufficiently small

$$h(0, y) = \begin{cases} t_0 - y - 1 & y > 0 \\ t_0 + y - 1 & y < 0 \end{cases}$$

and so $\frac{\partial h}{\partial y}$ does not exist at the origin. Hence $\tilde{\Phi}$ is not C^2 at the point $(z_0, 1)$, and by Proposition 4.2 the same must be true for Φ .

Theorem B. Let S be a finite union of points and curve segments in $\mathbb{P}^1 \setminus \{0\}$. Let $\phi \in C^\infty(\mathbb{P}^1)$ be a Kähler potential and suppose the Hele-Shaw flow for $\omega + dd^c \phi$ develops tangency along S . Then the weak solution Φ from (1) to the Dirichlet problem for the HMAE on $\mathbb{P}^1 \times \mathbb{D}$ with boundary data $(z, \tau) \mapsto \phi(\tau z)$ is not twice differentiable at the points $(\tau^{-1}z, \tau)$, $z \in S$, $|\tau| = 1$.

Proof of Theorem A and Theorem B. Suppose first that ϕ is as produced by Proposition 3.3. That is, we have picked points z_i in each component of $\mathbb{P}^1 \setminus \overline{\Omega_T}$ and $\pi: \Sigma \rightarrow \mathbb{P}^1 \setminus \{z_i\}$ is the universal cover, and $\Omega_{t \in (T-\epsilon, T]}$ is the pushforward of a strong Hele-Shaw flow on Σ . This implies that the normal velocity of the boundary of Ω_t as t tends to T from below is nowhere vanishing.

Now let $z \in S$, so Ω_T has boundary tangent to itself at z , and so Ω_T splits locally into two pieces, call them P_1 and P_2 . Working on P_1 , the combination of Proposition 5.3 (which says that $\partial\Omega_t$ are the level sets of $H(\cdot, 1) - 1$) and the fact that Ω_t varies smoothly

imply the partial derivative of $H(\cdot, 1)$ in the normal direction to Ω_t is strictly negative at z (see Example 6.1). Since the analogous statement is true for P_2 , this proves that H is not differentiable at $(z, 1)$. Thus $\tilde{\Phi}$ is not twice differentiable at the point $(z, 1)$, and by Proposition 4.2 the same is true for Φ . Then by S^1 -invariance we see that Φ cannot be twice differentiable at any point of the form $(\tau^{-1}z, \tau)$ for $z \in S, |\tau| = 1$.

Now if ϕ is any Kähler potential whose Hele-Shaw $\{\Omega_t\}$ develops tangency along S then it is not hard to see from the proof of Proposition 3.3 that $\{\Omega_t\}$ is the pushforward of some Hele-Shaw flow on Σ call it $\{\Omega'_t\}$. The hypothesis on Ω_T ensure that Ω'_T is smoothly bounded, and hence by the argument in Remark 2.12 we conclude that the normal velocity is non-vanishing as t tends to T from below (the reader who prefers not to invoke this argument may prefer to make this non-vanishing as part of the hypothesis of what it means to develop a tangency along S). The proof of the Theorem then follows as before.

Finally Theorem A follows from Theorem B and Proposition 3.3. \square

7. AN EXTENSION

So far we have been working under the hypothesis that our Hele-Shaw flow $\{\Omega_t\}_{t \in (0, V)}$ is standard as t tends to 0 (and also as t tends to 1 when $X = \mathbb{P}^1$). We did this to ensure regularity of the associated potential near the point 0 (resp. ∞) which we achieved by direct computation. In this section we explain how this hypothesis can be relaxed. For simplicity we work only with $(X, \omega) = (\mathbb{P}^1, \omega_{FS})$ but a similar story holds for the disc.

Definition 7.1. Let $\text{Diff}_0(\mathbb{P}^1)$ be the group of diffeomorphisms of \mathbb{P}^1 such that $\alpha(0) = 0$.

Given $\alpha \in \text{Diff}_0(\mathbb{P}^1)$ we define

$$\Omega_t = \alpha(B(t)) \text{ for } t \in (0, 1) \quad (14)$$

where, we recall, $B(t)$ denotes the geodesic ball centred at 0 with area t with respect to ω_{FS} . Clearly $\{\Omega_t\}_{t \in (0, 1)}$ is a strictly increasing, smoothly varying family of smoothly bounded simply connected domains in \mathbb{P}^1 that tends to zero as t tends to 0. We claim that for $\{\Omega_t\}_{t \in (0, 1)}$ constructed in this way the conclusion of Theorem C and Theorem 3.1 still hold; that is, there exists a Kähler potential ϕ such that $\{\Omega_t\}_{t \in (0, 1)}$ is the Hele-Shaw flow for ω_ϕ , and that weak geodesic obtained as the Legendre transform of the Hele-Shaw envelopes $\{\psi_t\}$ is regular.

We sketch why this is the case. Observe that the only place in which we used that $\{\Omega_t\}_{t \in (0, 1)}$ is standard as t tends to 0 and 1 in the proof of Theorem C was to ensure that ω_ϕ was a smooth Kähler form at 0 and at ∞ . So assume instead that (14) holds. Let $\alpha_0 = \text{id}_{\mathbb{P}^1} \in \text{Diff}_0(\mathbb{P}^1)$, whose associated flow is $\{B(t)\}_{t \in (0, 1)}$ which is the Hele-Shaw flow associated to ω_{FS} and, as we saw in (7), is the classical Hele-Shaw flow on \mathbb{C} with permeability $\kappa_0(z) := \pi(1 + |z|^2)^2$. Without loss of generality say $\alpha(z) = z + O(|z|^2)$ near $z = 0$. Then α is C^∞ close to α_0 in a neighbourhood of $z = 0$ which implies that Ω_t is C^∞ -close to $B(t)$ for t sufficiently small. In turn this implies the permeability κ as defined in (6) is C^∞ close to κ_0 in a punctured neighbourhood of 0, which is enough to imply it extends across 0 to a smooth strictly positive function. The argument near ∞ is similar: again without loss of generality say $\alpha(\infty) = \infty$ locally given by $\alpha(1/z) = 1/z + O(1/|z|^2)$ near $z = \infty$. Given a small neighbourhood U of ∞ we can construct an α_1 that is equal to α on $\mathbb{P}^1 \setminus U$ and is equal to α_0 near ∞ . Thus α is C^∞ close to α_1 and the same argument then applies to deduce that ϕ extends smoothly across ∞ and ω_ϕ is Kähler. Hence Theorem C still holds.

The argument for Theorem 3.1 is similar, as near $\{0\} \times \overline{\mathbb{D}}$ the foliation by harmonic discs provided by Proposition 5.4 for α is (in the obvious sense) C^∞ -close to that provided by

α_0 , and this is enough to prove that $\tilde{\Phi}$ is smooth over $\{0\} \times \overline{\mathbb{D}}$. Arguing similarly with α_1 near $\{\infty\} \times \overline{\mathbb{D}}$ we conclude that Theorem 3.1 still holds as well.

Accepting this argument, we see that to any $\alpha \in \text{Diff}_0(\mathbb{P}^1)$ we have an associated smooth geodesic ray in the space of Kähler metrics on \mathbb{P}^1 that starts at ω_ϕ and has limit $\omega + dd^c \ln |z|^2$ at infinity (i.e. as τ tends to zero). Of course different α can give rise to the same flow, but the ambiguity is precisely coming from the subgroup of “angular diffeomorphisms” given by

$$\Gamma := \{\alpha \in \text{Diff}_0(\mathbb{P}^1) : \alpha(B(t)) = B(t) \text{ for all } t\}.$$

Moreover this process can be reversed, since any smooth geodesic joining ω_ϕ to $\omega + dd^c \ln |z|^2$ comes from a regular solution to the complex Monge-Ampère Equation, and thus gives rise to a foliation by harmonic discs. By the harder direction of [19, Theorem 3.1] we know that such discs can only be those described in Proposition 5.4. Finally it is clear from the proof of Theorem 3.1 that different Hele-Shaw flows give rise to different potentials ϕ and vice versa. Thus in all we have the following explicit description of all smooth geodesics rays in the space of Kähler metrics on \mathbb{P}^1 that have limit $\omega + dd^c \log |z|^2$ as follows:

Theorem 7.2. *The duality that associates a weak geodesic ray to the Hele-Shaw flow gives a bijection between $\text{Diff}_0(\mathbb{P}^1)/\Gamma$ and*

$$\{\phi \in C^\infty(X) : \exists \text{ a smooth geodesic ray starting } \omega_\phi \text{ with limit } \omega_{FS} + dd^c \ln |z|^2\}.$$

Remark 7.3. Equivalently one can encode such a flow $\{\Omega_t\}$ by the smooth function on \mathbb{P}^1 whose level sets are $\partial\Omega_t$ (this is the approach taken in [6]).

APPENDIX A. SMOOTHNESS OF GREENS FUNCTIONS

We collect some regularity results for elliptic operators, all of which is essentially standard. Suppose $I \subset \mathbb{R}$ is an open interval and $\{L_t\}_{t \in I}$ is a smoothly varying family of strictly elliptic operators on the unit disc \mathbb{D} with uniform ellipticity constant. That is, we suppose

$$L_t u = a^{ij}(x, t) D_{ij} u + b^i(x, t) D_i u + c(x, t) u \text{ for } t \in I \quad (15)$$

where $a^{ij}, b^i, c \in C^\infty(\overline{\mathbb{D}} \times I)$ and u is a function defined on \mathbb{D} , such that there is a $\lambda > 0$ so that $a^{ij}(x, t) \zeta_i \zeta_j \geq \lambda |\zeta|^2$ for all $(x, t) \in \mathbb{D} \times I$ and $\zeta \in \mathbb{R}^N$. We assume also $c(x, t) \leq 0$ for $(x, t) \in \mathbb{D} \times I$.

Suppose now $\varphi \in C^\infty(\partial\mathbb{D} \times I)$, and we write $\varphi_t(\cdot) = \varphi(\cdot, t)$. Then for each $t \in I$ standard elliptic theory says [9, Corollary 6.9, Theorem 6.19] there exists a unique $u_t \in C^\infty(\overline{\mathbb{D}})$ that solves

$$L_t u_t = 0 \text{ and } u_t|_{\partial\mathbb{D}} = \varphi_t.$$

We claim that u_t is also smooth in the t -variable. To prove this it is sufficient to show it at $t = 0$ assuming $0 \in I$. Then expanding a^{ij}, b^i, c in t we can write

$$L_t u = L_0 u + t M_1 u + \cdots + t^N M_N u + O(t^{N+1}) u$$

for some operators M_i that are independent of t . Here and henceforth we work in the C^∞ -topology so the $O(t^{N+1})$ error terms means that for all $k \in \mathbb{N}$ there exists a C_k such that this term is bounded by $C_k |t|^{N+1}$ in the $C^k(\overline{\mathbb{D}})$ -norm. We wish to find an expansion for u_t in t , say

$$u_t = u_0 + t v_1 + \cdots + t^N v_N + O(t^{N+1}) \quad (16)$$

where $v_i \in C^\infty(\overline{\mathbb{D}})$. To do so expand $\varphi = \varphi_0 + t\sigma_1 + \cdots + t^N\sigma_N + O(t^{N+1})$ where $\sigma_i \in C^\infty(\partial\mathbb{D})$. Then comparing coefficients of t forces the v_i to satisfy

$$\begin{aligned} L_0 v_1 + M_1 u_0 &= 0 \text{ and } v_1|_{\partial\mathbb{D}} = \sigma_1 \\ L_0 v_2 + M_1 v_1 + M_2 u_0 &= 0 \text{ and } v_2|_{\partial\mathbb{D}} = \sigma_2 \end{aligned}$$

and so forth. So starting with u_0 we may inductively define v_i , and as L_0 is elliptic, the same elliptic regularity guarantees $v_i \in C^\infty(\overline{\mathbb{D}})$. To see that (16) does actually hold, observe that by construction the difference $w_t := u_t - u_0 - tv_1 - \cdots - v_N t^N$ satisfies

$$L_t w_t = O(t^{N+1}) \text{ and } w_t|_{\partial\mathbb{D}} = O(t^{N+1}).$$

Then, by elliptic theory again [9, Corollary 8.7, Theorem 8.13] this implies $w_t = O(t^{N+1})$ in the $C^\infty(\overline{\mathbb{D}})$ topology (here we are using that the elliptic constant for L_t is uniform over $t \in I$ to apply [9, Corollary 8.7] uniformly over I), which gives (16). As this holds for all N , the map $t \mapsto u_t$ is smooth in t , which implies $u \in C^\infty(\overline{\mathbb{D}} \times I)$ as claimed.

For the application we have in mind, suppose that $\{\Omega_t\}_{t \in I}$ is a smoothly varying family of smoothly bounded simply connected domains in \mathbb{D} containing 0. We may assume $\Omega_t = \alpha_t(\mathbb{D})$ where $\alpha: \overline{\mathbb{D}} \times I \rightarrow \mathbb{C}$ is smooth. Then set

$$L_t(u) := (\Delta(u \circ \alpha_t^{-1})) \circ \alpha_t$$

where u is a function on \mathbb{D} and Δ is the standard Laplacian on \mathbb{C} . Then $\{L_t\}_{t \in I}$ is a smoothly varying family of elliptic operators with uniform ellipticity constant, as in (15). Set $\Gamma(z) := \log|z|^2$ and $\varphi_t = \Gamma \circ \alpha_t$ so $\varphi \in C^\infty(\partial\mathbb{D} \times I)$. Then by the above discussion we know there exists a $u \in C^\infty(\overline{\mathbb{D}} \times I)$ such that

$$L_t u_t = 0 \text{ and } u_t|_{\partial\mathbb{D}} = \Gamma \circ \alpha_t.$$

Finally set

$$p_t := u_t \circ \alpha_t^{-1} - \Gamma$$

so by construction $\Delta p_t = 0$ on Ω_t and $p_t|_{\Omega_t} = 0$, that is p_t is minus the Greens function for Ω_t with logarithmic pole at 0. From this we see that p_t varies smoothly in t , in particular the quantity ∇p_t on $\partial\Omega_t$ is a smooth vector field on $\cup_{t \in I} \Omega_t$ which is precisely what we used in the proof of Theorem 3.1.

REFERENCES

- [1] C. Arezzo and G. Tian *Infinite geodesic rays in the space of Kähler potentials* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 2 (2003), no. 4, 617–630
- [2] E. Bedford and J.-P. Demailly *Two counterexamples concerning the pluri-complex Green function in \mathbb{C}^n* . Indiana Univ. Math. J. 37 (1988), no. 4, 865–867.
- [3] Z. Błocki *The $C^{1,1}$ regularity of the pluricomplex Green function* Michigan Math. J. 47 (2000), 211–215.
- [4] Z. Błocki *On geodesics in the space of Kähler metrics* Advances in geometric analysis, 3719, Adv. Lect. Math. (ALM), 21, Int. Press, Somerville, MA, 2012.
- [5] X. Chen *The space of Kähler metrics* J. Differential Geom. 56 (2000), no. 2, 189–234.
- [6] S. Donaldson *Symmetric spaces, Kähler geometry and Hamiltonian dynamics* Northern California Symplectic Geometry Seminar, 13–33, Amer. Math. Soc. Transl. Ser. 2, 196, Amer. Math. Soc., Providence, RI, 1999
- [7] S. Donaldson *Holomorphic discs and the complex Monge-Ampère equation*. J. Symplectic Geom. 1 (2002), no. 2, 171–196.
- [8] B. Gustafsson and A. Vasil’ev *Conformal and potential analysis in Hele-Shaw cells* Advances in Mathematical Fluid Mechanics. Birkhäuser Verlag, Basel, 2006. x+231
- [9] D. Gilbarg and N. Trudiger *Elliptic partial differential equations of second order* Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001. xiv+517 pp. ISBN: 3-540-41160-7
- [10] H. Hedenmalm and S. Shimorin *Hele-Shaw flow on hyperbolic surfaces* J. Math. Pures Appl. (9) 81 (2002), no. 3, 187–222.

- [11] S. Krantz *Function theory of several complex variables* Reprint of the 1992 edition. AMS Chelsea Publishing, Providence, RI, 2001. xvi+564 pp. ISBN:
- [12] L. Lempert *La métrique de Kobayashi et la représentation des domaines sur la boule* Bull. Soc. Math. France 109 (1981), no. 4, 427–474.
- [13] L. Lempert and L. Vivas *Geodesics in the space of Kähler metrics* Duke Math. J. 162 (2013), no. 7, 1369–1381
- [14] L. Lempert and T. Darvas *Weak geodesics in the space of Kähler metrics* Math. Res. Lett. 19 (2012), no. 5, 1127–1135.
- [15] T. Mabuchi *Some symplectic geometry on compact Kähler manifolds. I.* Osaka J. Math. 24 (1987), no. 2, 227–252
- [16] S. Richardson *Hele-Shaw flows with a free boundary produced by the injection of fluid into a narrow channel* J. Fluid Mech., 56 (1972), no. 4, 609–618.
- [17] J. Ross and D. Witt Nyström *Homogeneous Monge-Ampère Equations and Canonical Tubular Neighbourhoods in Kähler Geometry* (2014) Preprint arXiv:1403.3282
- [18] J. Ross and D. Witt Nyström *The Hele-Shaw flow and Moduli of Holomorphic Discs* To appear in Compositio Mathematica
- [19] J. Ross and D. Witt Nyström *Harmonic Discs of Solutions to the Complex Homogeneous Monge-Ampère Equation* To appear in Publ. Math. IHES.
- [20] Y. Rubinstein and S. Zelditch *The Cauchy problem for the homogeneous Monge-Ampère equation, I. Toeplitz quantization.* J. Differential Geom. 90 (2012), no. 2, 303–327.
- [21] Y. Rubinstein and S. Zelditch *The Cauchy problem for the homogeneous Monge-Ampère equation, II. Legendre transform.* Adv. Math. 228 (2011), no. 6, 2989–3025.
- [22] S. Semmes *Complex Monge-Ampère and symplectic manifolds* Amer. J. Math. 114 (1992), no. 3, 495–550.

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